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2D Ising Model with Layers of Quenched Spins

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The model considered is a $d=2$ disordered Ising system on a square lattice with nearest neighbor interaction. The disorder is induced by layers (rows) of spins, randomly located, which are frozen in an antiferromagnetic order. It is assumed that all the vertical couplings take the same positive value J_v, while all the horizontal couplings take the same positive value J_h. The model can be exactly solved and the free energy is given as a simple explicit expression. The zero-temperature entropy can be positive because of the frustration due to the competition between antiferromagnetic alignment induced by the quenched layers and ferromagnetic alignment due to the positive couplings. No phase transition is found at finite temperature if the layers of frozen spins are independently distributed, while for correlated disorder one finds a low-temperature phase with some glassy properties.

KEY WORDS: Disorder; Ising systems; quenched average; spin glasses; Onsager solution.

1. INTRODUCTION

The exact solution of the S.K. model[1] has permitted a deep understanding of the nature of the low temperature phase of Ising spin glasses[2] and has introduced many new concepts like replica symmetry breaking, overlap distribution function and ultrametricity. Unfortunately, most of the typical features of the mean field models have not been established for short range spin glasses.[3–13] For example, the existence of a glassy phase for $d \geqslant 3$ is widely accepted but it is not clear if this phase is qualitatively the same of that of the S.K. model. Furthermore, for $d=2$ it is commonly believed that only the paramagnetic phase is present, which is true for $d=2$ spin system

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with independent bonds and with vertical-horizontal symmetry but may be not true for $d=2$ spin systems with correlated disorder or asymmetric.[14] In particular, the existence of a low temperature phase in systems with frustration induced by layered disorder seems to be an established fact.[15, 16] The reason why a complete understanding of short range systems is still missing is that it is very difficult to handle this model from an analytic point of view. Indeed, in presence of disorder even a $d=1$ system with magnetic field is a very complicated problem[17–25] and explicit exact solutions can be found only in special cases.[26]

In this paper I try to give a contribution to the understanding of short range spin glasses by exactly and explicitly computing the free energy of $d=2$ random layered Ising system. The model is defined as follows: the interaction is effective only between nearest neighbours on a square lattice; all the vertical couplings take the same positive value J_v while all the horizontal couplings take the same positive value J_h, some horizontal layers (rows) of spins, randomly located, are frozen in an antiferromagnetic order. This model is, indeed, very simple since it is an ordinary asymmetric Ising system where the spins of some rows are not free to arrange themselves according with a Gibbs measure but are quenched variables. The exact solution is found by dialing with the row to row random transfer matrices following the line of.[27] This approach allows a decomposition of the transfer matrices in a direct product of 2×2 random matrices. An important technical step is then the explicit computation of the traces of products of these 2×2 matrices. It is found out that the zero temperature entropy is positive for some choices of the coupling strengths; this is a consequence of the competition between antiferromagnetic alinement induced by the quenched layers and ferromagnetic alinement due to the positive couplings which induces a strong frustration in the system. Surprisingly, the existence of a phase transition depends on the distribution of the quenched layers. In the simplest case it is assumed that a row of spin is frozen with probability p independently from the others. For this uncorrelated disorder there is not no phase transition at finite temperature. On the contrary, for correlated disorder one finds a low temperature phase with some glassy properties. The nature of this non-ferromagnetic low temperature phase is still unclear, but there are some indications that it shares some of the properties of a glassy phase.

Before ending this section I would like to mention that layered Ising models were first considered by B. M. McCoy and T. T. Wu[28, 29] in a non frustrated context which was proposed for studying the effect of quenched randomness on the ferro-para transition. These authors deal with the determinant which occurs in the Pfaffian approach and, while they do not provide an explicit exact solution of the problem, they are able to show that the free energy has an infinitely differentiable singularity at the transition. Layered models with frustration have been also studied by R. Shankar and Ganpathy Murthy,[16] not only their topic but also their approach is the same of this work since they deal with the row to row transfer matrices. They do not find out an exact solution, nevertheless they map the problem into a collection of $d=1$ random field Ising systems from which they can extract a lot of informations. In particular they provide evidence for the existence of a low temperature phase.

2. THE MODEL

Let me now state the problem. Assume that $N=LM$ is the number of spins, L is the number of rows and M the number of columns, the hamiltonian can be simply written as

$$H=-\sum_{ij}\left(J_v\sigma_{i,j}\sigma_{i+1,j}+J_h\sigma_{i,j}\sigma_{i,j+1}\right) \tag{2.1}$$

where the $\sigma_{i,j}$ are dichotomic ± 1 variables located at the i-row and j-column. Manifestly, this way of writing down the Hamiltonian is misleading since the spins of some of the horizontal rows are quenched variables frozen in an antiferromagnetic order. In order to write down the above hamiltonian in a more explicit form let me introduce the quenched variables η_i which can take the values 0 and 1 according to a given distribution. The meaning of this variable is the following, $\eta_i=0$ corresponds to a i-row of ordinary spins while $\eta_i=1$ corresponds to a i-row of quenched spins. In the independent case $\eta_i=0$ with probability $1-p$ and $\eta_i=1$ with probability p.

Let me than introduce the hamiltonian

$$H(K)=-\sum_{ij}\left[J_v\sigma_{i,j}\sigma_{i+1,j}+J_h\sigma_{i,j}\sigma_{i,j+1}-\eta_i K(1+\sigma_{i,j}\sigma_{i,j+1})\right] \tag{2.2}$$

if the limit $K\to\infty$ is performed one obtains that all the spins of the rows with $\eta_i=1$ are frozen in an antiferromagnetic order. In conclusion the hamiltonian (2.1) can be rewritten as $\lim_{K\to\infty} H(K)$.

The partition function associated to (2.2) is

$$Z(K)=\sum_{\{\sigma\}}\exp\left\{\sum_{ij}\beta(J_v\sigma_{i,j}\sigma_{i+1,j}+J_h\sigma_{i,j}\sigma_{i,j+1}-\eta_i K(1+\sigma_{i,j}\sigma_{i,j+1}))\right\} \tag{2.3}$$

After having defined $\Gamma_h = \beta J_h$ and $\Gamma_v \equiv \beta J_v$ and having performed the limit $K \to \infty$ one obtains

$$Z = \sum_{\{\sigma\}} \prod_{ij} \left[\exp\{\Gamma_v \sigma_{i,j} \sigma_{i+1,j} + \Gamma_h \sigma_{i,j} \sigma_{i,j+1}\} \left(1 - \frac{1 + \sigma_{i,j} \sigma_{i,j+1}}{2} \eta_i \right) \right] \tag{2.4}$$

The terms in parenthesis equal 1 when $\eta_i = 0$ and $(1 - \sigma_{i,j} \sigma_{i,j+1})/2$ when $\eta_i = 1$. Notice that in this second case the antiferromagnetic order between neighbour spins on the row is imposed, in fact, if $\sigma_{i,j}$ and $\sigma_{i,j+1}$ have the same sign they give a vanishing contribution to the partition function.

The free energy to be computed is

$$f = - \lim_{N \to \infty} \frac{1}{\beta N} \log Z \tag{2.5}$$

which can be also obtained as the $K \to \infty$ limit of the free energy $f(K)$ associated to the partition function (2.3)

$$f(K) = \lim_{N \to \infty} - \frac{1}{\beta N} \log Z(K) \tag{2.6}$$

It should be noticed that the frustration comes out from the fact that the tendency to the ferromagnetic alinement, due to the positive couplings, is in competition with the tendency to the antiferromagnetic alinement induced by the frozen spins on the unfrozen ones. A somehow correlated problem, where the spin are randomly frozen in a random direction has been solved in $d = 1$ in ref. 26, and studied in $d = 2$ at zero temperature in ref. 30.

3. THE TRANSFER MATRICES APPROACH

The advantage of considering layered disorder is that one can follow the standard method of Schultz, Mattis and Lieb,[27] and reduce the problem to the evaluation of the trace of products of 2×2 random matrices.

The first step is to rewrite the partition function as

$$Z(K) = \mathrm{Tr} \prod_{i=1}^{L} V_i(K) \tag{3.1}$$

where the $V_i(K)$ are $2^M \times 2^M$ row to row transfer matrices which can be expressed in terms of the quantities $\Gamma_i \equiv \Gamma_h - K\beta\eta_i$ and $\Gamma_v^* \equiv -\frac{1}{2}\log(\tanh \Gamma_v)$ in the following way

$$V_i(K) = (2\sinh 2\Gamma_v)^{M/2} \exp\left\{\sum_j \Gamma_v^* \tau_j^z\right\} \exp\left\{\sum_j (\Gamma_i \tau_j^x \tau_{j+1}^x - \eta_i K\beta)\right\} \quad (3.2)$$

where the $2^M \times 2^M$ matrices τ_j^x and τ_j^z are obtained as the direct product of $M-1$ 2×2 identity matrices and, respectively, of the Pauli matrices τ^x and τ^z placed in the j "position."

The method of Schultz, Mattis and Lieb[27] allows a decomposition of the transfer matrices $V_i(K)$ in a direct product of 2×2 matrices as follows:

$$V_i(K) = (2\sinh 2\Gamma_v)^{M/2} \otimes_q T_i(q, K) \quad (3.3)$$

where $\otimes_q$ indicates the direct product over $q = 0$, $(2\pi/M)$, $(4\pi/M)$,..., $((M-2)\pi)/M$, π and the 2×2 matrices

$$T_i(q, K) = \exp\{2\Gamma_v^*(\tau^z \cos q + \tau^x \sin q)\} \exp\{-2\Gamma_i \tau^z\} \exp\{-2\beta K\eta_i\} \quad (3.4)$$

are written in terms of the Pauli matrices τ^x and τ^z.

Taking into account that $\Gamma_i = \Gamma_h - \beta K\eta_i$ one can also write (3.4) as

$$T_i(q, K) = T(q)\, E_i(K) \quad (3.5)$$

where $T(q)$ is independent on K and non random

$$T(q) = \exp\{2\Gamma_v^*(\tau^z \cos q + \tau^x \sin q)\} \exp\{-2\Gamma_h \tau^z\} \quad (3.6)$$

while $E_i(K)$ depends both on K and η_i

$$E_i(K) = \exp\{2\beta K\eta_i(\tau^z - 1)\} \quad (3.7)$$

Following the same steps of ref. 27, it is now a simple exercise to perform the limit $M \to \infty$ and to find from (2.6) and (3.1) the free energy

$$f(K) = -\frac{1}{2\beta}\log(2\sinh 2\Gamma_v) - \frac{1}{2\pi\beta}\int_0^\pi \gamma(q, K)\, dq \quad (3.8)$$

where $\gamma(q, K)$ is given in terms of a product of random 2×2 matrices as

$$\gamma(q, K) = \lim_{L\to\infty} \frac{1}{L} \log \operatorname{Tr} \prod_{i=1}^{L} [T(q)\, E_i(K)] \quad (3.9)$$

This is the point where exact computations in this kind of problems usually end since the trace of a product of random matrices can be computed numerically and well approximated analytically but it has not, in general, an explicit compact expression. For this model, on the contrary, the exact solution is found out. This is possible because in the limit $K \to \infty$ so the trace of the product of transfer matrices (3.9) becomes exactly computable.

Before starting the next section let me notice that since τ^z commutes with $E_i(K)$ (3.9) can be also written as

$$\gamma(q, K) = \lim_{L \to \infty} \frac{1}{L} \log \operatorname{Tr} \prod_{i=1}^{L} [\tilde{T}(q)\, E_i(K)] \tag{3.10}$$

where

$$\tilde{T}(q) = \exp\{-\Gamma_h \tau^z\} \exp\{2\Gamma_v^*(\tau^z \cos q + \tau^x \sin q)\} \exp\{-\Gamma_h \tau^z\} \tag{3.11}$$

4. THE SOLUTION

In the $K \to \infty$ limit one has $E_i(K) \to E_i$

$$E_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 - \eta_i \end{pmatrix} \tag{4.1}$$

$$f = -\frac{1}{2\beta} \log(2 \sinh 2\Gamma_v) - \frac{1}{2\pi\beta} \int_0^\pi \gamma(q)\, dq \tag{4.2}$$

$$\gamma(q) = \lim_{L \to \infty} \frac{1}{L} \log \operatorname{Tr} \prod_{i=1}^{L} [\tilde{T}(q)\, E_i] \tag{4.3}$$

where $\tilde{T}(q)$ is defined by (3.11). The trace of a product of random matrices is easily accessible via computer simulation but it cannot be, in general, exactly computed. In the present case, nevertheless, following a similar method as in ref. 26, it is possible to find out the compact analytical result. Consider a given realization of the quenched variables η_i, the product of matrices in (4.3) is a product of matrices $\tilde{T}(q)$ and up projectors τ^+. The first and the second τ^+ will be separated by l_1 matrices $\tilde{T}(q)$, the second and the third by l_2 matrices $\tilde{T}(q)$, and so on. The l_n are random variable which can take the values 1, 2,... whose distribution can be easily found out once the distribution of the η_i is given. The order number n goes from 1 to

$n_f = L/\bar{l}$, in fact, one must have $\sum_{n=1}^{n_f} l_n = L$ so that $\sum_{n=1}^{n_f} l_n/n_f \equiv \bar{l} = L/n_f$. With the help of these considerations one can rewrite (4.3) as

$$\gamma(q) = \lim_{L \to \infty} \frac{1}{L} \log \prod_{n=1}^{L/\bar{l}} [\tilde{T}(q)^{l_n}]_{11} = \lim_{L \to \infty} \frac{1}{L} \sum_{n=1}^{L/\bar{l}} \log[\tilde{T}(q)^{l_n}]_{11} \tag{4.4}$$

where $[\tilde{T}(q)^{l_n}]_{11}$ is the up left entry of $\tilde{T}(q)^{l_n}$. It must be noticed that the random numbers l_n have a simple geometrical meaning, in fact, the distances between two successive rows of frozen spins are $l_n - 1$. Furthermore, the rate of frozen layers is $1/\bar{l}$ and the rate of unfrozen ones is $(\bar{l} - 1)/\bar{l}$. If $P(l)$ is the probability distribution of the l_n than $\bar{l} \equiv \sum_{l=1}^{\infty} lP(l)$ and

$$\gamma(q) = \sum_{l=1}^{\infty} \frac{1}{\bar{l}} P(l) \log[\tilde{T}(q)^l]_{11} \tag{4.5}$$

In order to find the explicit form for (4.5) it is convenient to define

$$c \equiv \cosh 2\Gamma_h, \quad c^* \equiv \cosh 2\Gamma_v^*, \quad s \equiv \sinh 2\Gamma_h, \quad s^* \equiv \sinh 2\Gamma_v^* \tag{4.6}$$

and

$$\cosh \varepsilon \equiv cc^* - ss^* \cos q, \quad \cos \phi \equiv \frac{(cs^* \cos q - sc^*)}{((s^* \sin q)^2 + (cs^* \cos q - sc^*)^2)^{1/2}} \tag{4.7}$$

With these definitions it is immediate to rewrite $\tilde{T}(q)$ in the simpler form

$$\tilde{T}(q) = \exp\{\varepsilon(\tau_3 \cos \phi + \tau_1 \sin \phi)\} \tag{4.8}$$

which immediately gives

$$[\tilde{T}(q)^l]_{11} = \cosh(l\varepsilon) + \cos \phi \sinh(l\varepsilon) \tag{4.9}$$

Finally, collecting (4.2), (4.5) and (4.9) one has

$$\begin{aligned} f = &-\frac{1}{2\beta} \log(2 \sinh 2\Gamma_v) \\ &- \frac{1}{2\pi\beta\bar{l}} \sum_{l=1}^{\infty} P(l) \int_0^{\pi} \log(\cosh(l\varepsilon) + \cos \phi \sinh(l\varepsilon))\, dq \end{aligned} \tag{4.10}$$

which is the wanted explicit and compact expression for the free energy. It should be noticed, that (4.10) is the sum of the free energies of strips of side l, i.e., strips of $l - 1$ unfrozen layers between two frozen ones. This decomposition is a consequence of the fact that frozen layers decouple the strips.

5. INDEPENDENT LAYERS

The free energy computed in the previous section depends on the choice of $P(l)$ which is the probability that two frozen layers are separated by $l-1$ unfrozen layers. Different choices corresponds to qualitatively different physical behaviours of the system and the existence itself of low phase temperature strictly depends on $P(l)$.

The simplest distribution corresponds to independent quenched layers, which means that any layer is frozen with probability p independently from the others (i.e., $\eta_i = 1$ with probability p). In this case the probability $P(l)$ has the exponential form

$$P(l) = p(1-p)^{l-1} \tag{5.1}$$

and $\bar{l} = 1/p$.

With this choice the system has no ordinary phase transition, except for $p = 0$ where it trivially reduces to the ordinary Ising model (no frozen layers). In order to understand this fact, let me stress again that (4.10) is the sum of the free energies of one-dimensional strips of side l. Far from the Onsager critical temperature T_c the contribution of a l-strip to the free energy (4.10) (and any derivative of it) is proportional to l. Since the weight $P(l)$ is exponentially small with l, the largest contribution to the thermodynamical quantities comes from strips of finite side length. Furthermore, by expanding the free energy around critical temperature, one see that at T_c the contribution of a l-strip to the n-derivative of the internal energy is of order l^n. Again, since the weight is exponentially small in l, the n-derivative of the sum (4.10) cannot diverge at T_c. In conclusion, there are not discontinuous thermodynamical quantities and there is not an ordinary phase transition at T_c.

Nevertheless, it should be noticed that while at T_c the relevant contribution comes from finite l-strips, the larger is n, the larger is the typical l. This fact implies that the n-derivative of the internal energy diverges at T_c when n goes to infinity. This is a consequence of the fact that there is a finite probability of having strips with very large l, which are arbitrarily close to a critical behaviour. This scenario is very similar to the one described in ref. 28.

In Fig. 1 it is shown the specific heat C in correspondence of different values of p. When $p = 0$ one has the logarithmic divergence of the ordinary Ising model, when p increases ($p = 0.05$, 0.1, 0.2) one can notice that the logarithmic divergence is smoothed. Further derivatives of the specific heat also have the same smooth behaviour for $p \neq 0$ at all temperatures showing the absence of transition.

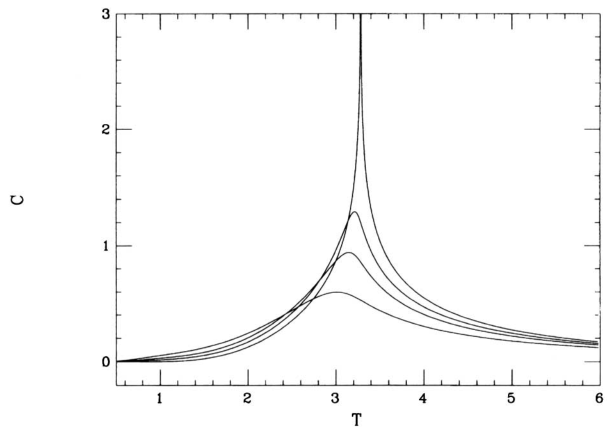

Fig. 1. Specific heat C as function of the temperature T for the $P(l) = p(1-p)^{l-1}$ case. The couplings are $J_h = 1$ and $J_v = 2$. The singular line corresponds to the Ising model ($p = 0$), the other lines to $p = 0.05$, $p = 0.1$ and $p = 0.2$.

Since the transition disappears for $p \neq 0$ the role of p reminds that of a magnetic field which also suppresses the transition. The analogue of the spontaneous magnetization is than obtained as the derivative of the free energy with respect to p in correspondence of $p = 0$ (i.e., it is given by the limit $p \to 0$ of $f' \equiv [\partial f/\partial p]_{p=0}$).

In order to compute this quantity, it is useful to notice that when p becomes very small, only very large values of l contribute to the sum in (4.10). For large values of l one can write

$$\cosh(l\varepsilon) + \cos\phi \sinh(l\varepsilon) = e^{l\varepsilon}\left(\frac{1+\cos\phi}{2}\right)(1 + O(e^{-2l\varepsilon})) \tag{5.2}$$

Using this approximate expression in (4.10) it is very easy to obtain

$$f' \equiv \left[\frac{\partial f}{\partial p}\right]_{p=0} = -\frac{J}{2\pi\Gamma}\int_0^\pi \log\left(\frac{1+\cos\phi}{2}\right) dq \tag{5.3}$$

As mentioned, (5.3) is somehow an analogous of the spontaneous magnetization, and as well as the spontaneous magnetization, this quantity is continuous while its derivative df'/dT is not. The derivative df'/dT for different values of the parameters is shown in Fig. 2 where one can see a logarithmic divergence at T_c (the Onsager critical temperature).

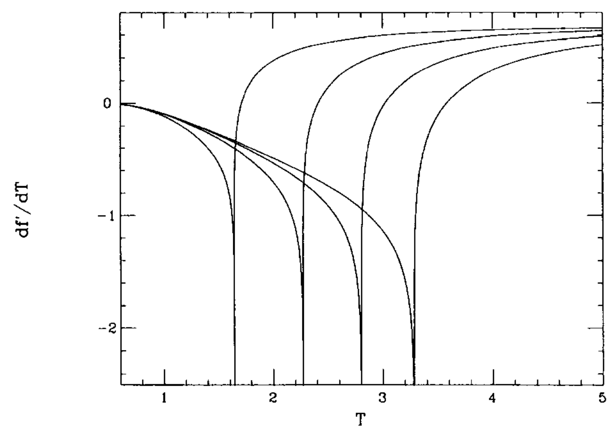

Fig. 2. Temperature derivative df'/dT of $f' \equiv [\partial f/\partial p]_{p=0}$ as function of T. The horizontal coupling is $J_h = 1$, the different lines correspond to the vertical couplings $J_v = 0.5$, $J_v = 1$, $J_v = 1.5$ and $J_v = 2$.

6. CORRELATED DISORDER

Uncorrelated disorder implies that the system as no phase transition, therefore, the question is: are there choices of $P(l)$ which correspond to a two phases behaviour of the system? The answer is positive and intuition says that one as to look for correlated distribution with higher probability of longer distances between quenched layers. In fact, in this case, very large strips may have a sufficiently large probability to influence the behaviour of the system at the critical temperature.

We have already seen that the n-derivative of the internal energy goes as l^n at the transition. This fact suggest to consider a power law $P(l)$ as, for example,

$$P(l) = a/(l+q)^3 \tag{6.1}$$

where a is the normalization constant and q is a parameter which can assume any value $q > -1$. Notice that the larger is q, the larger is the average distance between frozen layers. The ordinary Ising mode is than recovered in the limit $q \to \infty$. Also notice that for the above power low probability, which replaces the exponential $P(l) = p(1-p)^{l-1}$ of the independent case, one has $\overline{l^x} = \infty$ for $x \geqslant 2$.

By substituting (6.1) in (4.10) one discovers that there is phase transition at the Onsager temperature. Nevertheless, this phase transition does not correspond to a divergence in the specific heat C. In fact, in Fig. 3

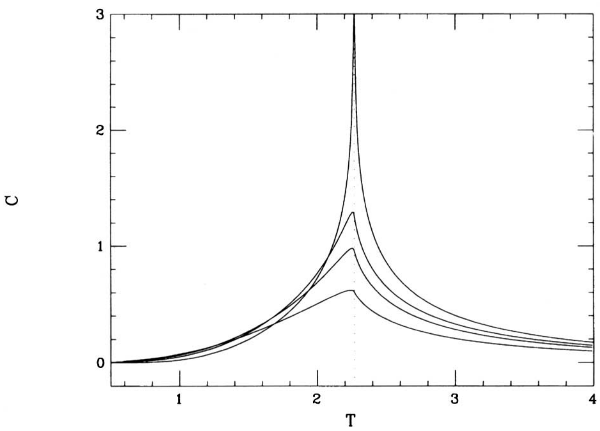

Fig. 3. Specific heat C as function of the temperature T for the $P(l) = a/(l+q)^3$ case. The couplings are $J_h = 1$ and $J_v = 1$. The singular line corresponds to the Ising model ($q = \infty$), the other lines to $q = 10$, $q = 5$ and $q = 2$. The dots indicate the Onsager critical temperature.

C is plotted for different values of q, the divergent line corresponds to the Onsager case $q = \infty$, while the progressively smoothed lines correspond to $q = 10$, 5, 2. Apparently this figure is qualitatively the same as Fig. 1, but there is a fundamental difference: at the Onsager temperature (dots) all the lines have a divergent derivative. For this reason dC/dT has been plotted for the same choice of the parameters in Fig. 4 which clearly shows that this quantity is divergent at the critical temperature.

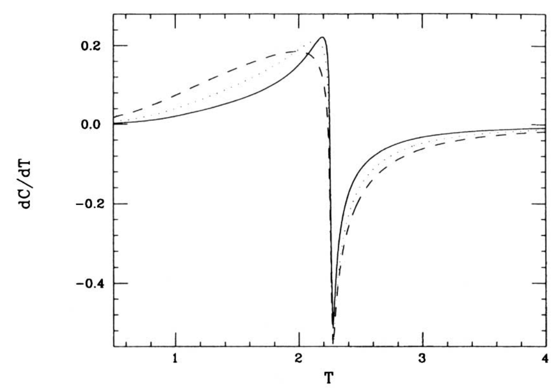

Fig. 4. Temperature derivative $(1/\bar{l})(dC/dT)$ of the specific heat function of the temperature T for the $P(l) = a/(l+q)^3$ case. The couplings are $J_h = 1$ and $J_v = 1$. The values of the parameter are $q = 10$ (full), $q = 5$ (dots), $q = 2$ (dash).

Power laws with larger exponents would have divergencies only at higher derivatives of the specific heat, while for much less correlated layers (for example with a stretched exponential distribution) one would have a behaviour qualitatively similar to the independent case.

I have not been able to quantitatively characterize the low phase temperature with an order parameter. In fact, while the spontaneous magnetization $\overline{\langle \sigma_{i,j} \rangle}$ vanishes (the bar means average over all realizations of the disorder), both the overlap $\overline{\langle \sigma_{i,j} \rangle^2}$ and the correlation between frozen and unfrozen spins on the same column differ from zero at any temperature. I think that a probable candidate for the order parameter is the exponent of the horizontal long range correlation between unfrozen spins $\overline{\langle \sigma_{i,j} \sigma_{i,j+k} \rangle} - \overline{\langle \sigma_{i,j} \rangle^2} \approx e^{-\lambda k}$. I argue, in fact, that the decay exponent λ of this correlation have discontinuous derivative at the transition point. Unfortunately, this quantity seems to be very difficult to compute if one follows the standard procedure of ref. 27 so that some new method is probably needed.

What is easy to compute is the first neighbour horizontal correlation cr between unfrozen spin. Taking into account that $(\bar{l}-1)\, L/\bar{l}$ is the number of unfrozen layers and $(\bar{l}-1)\, N/\bar{l}$ is the number of unfrozen spins, this quantity can be defined as

$$cr \equiv \lim_{N \to \infty} \frac{\bar{l}}{(\bar{l}-1)\, N} \sum_{i,j} \langle \sigma_{i,j} \sigma_{i,j+1} \rangle \tag{6.2}$$

were the sum goes over all nearest horizontal couples of spins of the unfrozen layers (over all the j and over the i corresponding to $\eta_i = 0$). This quantity is manifestly self-averaging so that one can also write

$$cr = \overline{\langle \sigma_{i,j} \sigma_{i,j+1} \rangle} \tag{6.3}$$

From a practical point of view it is easy to obtain cr from the free energy, as

$$c = -\frac{\bar{l}}{(\bar{l}-1)} \left(\frac{\partial f}{\partial J_h} - \frac{1}{\bar{l}} \right) \tag{6.4}$$

this formula is easy to understand if one remember that the fraction of frozen layers is $1/\bar{l}$ and their correlation is totally negative. In Fig. 5 cr is plotted for two very close values of the horizontal coupling. Notice that the behaviour of cr is quite complicated, in fact, this quantity is far from being monotonical and it is positive (ferro) in some ranges of the temperature and negative (antiferro) in others. Furthermore, approaching the zero temperature the two correlations, which correspond to very similar couplings,

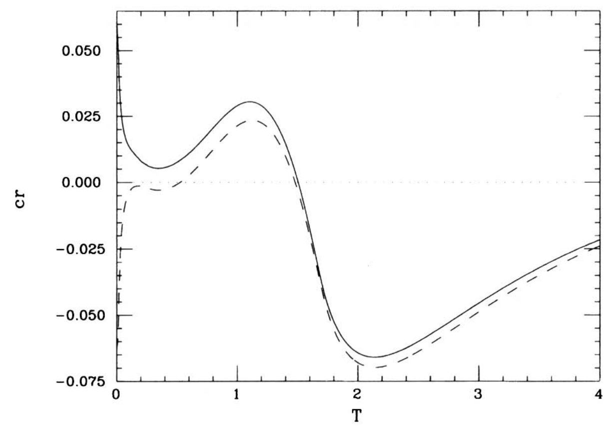

Fig. 5. Horizontal first neighbors correlation $cr = \overline{\langle \sigma_{i,j} \sigma_{i,j+1} \rangle}$ for the $P(l) = a/(l+q)^3$ case. The parameter is $q = 1$, the horizontal coupling is $J_h = 0.3$, the vertical couplings are $J_v = 1.49$ (full) and $J_v = 1.51$ (dash).

split and they end at zero temperature in quite far points. This last fact is a consequence of the frustration of the model which implies a non trivial zero temperature behaviour of the thermodynamical quantities as it is shown in the next section.

7. ZERO TEMPERATURE

Because of the competition between the tendency to ferromagnetic alinement induced by the positive couplings and the tendency to antiferromagnetism induced by the frozen layers, the model is frustrated and its zero temperature properties are not completely trivial. In order to compute the free energy and its derivatives at $T = 0$ it is useful to rewrite the expressions in (4.7) in the approximate form

$$\varepsilon \simeq 2\beta J_h, \qquad \cos\phi \simeq -1 + 8\sin^2 q e^{-4\beta(J_h + J_v)} \tag{7.1}$$

which differ from the exact values only by exponentially small in T quantities. In the same approximation, using (7.1), is than easy to obtain from (4.10)

$$f \simeq -\left(1 - \frac{2}{\bar{l}}\right)(J_v + J_h) - \sum_{l=1}^{\infty} \frac{P(l)}{2\pi\beta\bar{l}} \int_0^{\pi} \log[(2\sin q)^2 + e^{-4\beta[(l-1)J_h - J_v]}]\, dq \tag{7.2}$$

From this approximated free energy one can compute the $T=0$ exact energy f_0 end entropy s_0. Assume that $(n-1)\,J_h < J_v \leqslant nJ_h$ where n is a positive integer, then

$$f_0 = -\left(1-\frac{2}{\bar{l}}\right)(J_v + J_h) + \frac{2}{\bar{l}} \sum_{l=1}^{n} P(l)[(l-1)\,J_h - J_v] \tag{7.3}$$

which is continuous with respect to the couplings, but with discontinuous derivatives. The $T=0$ entropy can be also easily computed and one finds that it vanishes when for $J_v \neq nJ_h$ for any integer n. On the contrary

$$s_0 = \frac{P(n+1)}{\bar{l}} \log\left(\frac{\sqrt{5}+1}{2}\right) \tag{7.4}$$

for $J_v = nJ_h$ showing an exponential degeneration of the ground state. The reason why s_0 is not vanishing is that in this case the frustration is maximal since alinement following the frozen layers or nearest unfrozen spins becomes energetically equivalent.

Figure 5 shows that something also has to happen at $T=0$ to the correlation cr. From (7.2), using (6.4), one finds that

$$cr_0 = 1 - \frac{2}{\bar{l}} - \frac{2}{\bar{l}} \sum_{l=1}^{n} P(l)(l-1) \tag{7.5}$$

for $(n-1)\,J_h < J_v < nJ_h$. This formula implies that the correlation has a jump where the couplings are commensurate. In fact, if one goes from $J_v = nJ_h - \varepsilon$ which is in the range $(n-1)\,J_h < J_v < nJ_h$ to $J_v = nJ_h + \varepsilon$ which is in the range $nJ_h < J_v < (n+1)\,J_h$, then cr_0 decreases from the value (7.5) to

$$cr_0 = 1 - \frac{2}{\bar{l}} - \frac{2}{\bar{l}} \sum_{l=1}^{n} P(l)(l-1) - \frac{2}{\bar{l}} P(n+1)\,n \tag{7.6}$$

This is the situation illustrated in Fig. 5 where $J_h = 0.3$ and J_v takes the two values 1.49 and 1.51 which are just a little smaller and a little larger of the commensurate value $1.5 = 5J_h$. Moreover, from (7.2) one also finds that at the discontinuity point $J_v = nJ_h$ one exactly has

$$cr_0 = 1 - \frac{2}{\bar{l}} - \frac{2}{\bar{l}} \sum_{l=1}^{n} P(l)(l-1) - \frac{2}{\sqrt{5}\bar{l}} P(n+1)\,n \tag{7.7}$$

which is intermediate between the two values (7.5) and (7.6) of cr_0 at the two sides of the transition point.

This complex behavior at vanishing temperature, which is typical of glassy systems, should not completely disappear in the low temperature phase which also should have some glassy properties. For example, since (7.5) implies an exponential degeneration of the ground state, one may expect that at low temperatures the relevant contributions to the free energy comes from macroscopically separated spin configurations.

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REFERENCES

1. D. Sherrington and S. Kirkpatrick, *Phys. Rev. Lett.* **32**:1792 (1975).
2. M. Mezard, G. Parisi, and M. Virasoro, *Spin glass theory and beyond* (World Scientific Singapore, 1988).
3. M. E. Fisher and W. Selke, *Phys. Rev. Lett.* **44**:1502 (1980).
4. F. Koukiou, *Europhys. Lett.* **7**:297 (1992).
5. L. Saul and M. Kardar, *Phys. Rev. E* **48**:48 (1993).
6. J. L. Van Hemmen and R. G. Palmer, *J. Phys. A: Math. Gen.* **15**:3881 (1982).
7. H.-F. Cheung and W. McMillan, *J. Phys. C* **11**:7027 (1983).
8. G. Paladin, M. Pasquini, and M. Serva, *Int. J. Mod. Phys. B* **9**:399 (1995).
9. G. Paladin, M. Pasquini, and M. Serva, *J. Phys. I (France)* **5**:337 (1995).
10. S. Scarlatti, M. Serva, and M. Pasquini, *J. Stat. Phys.* **80**:337 (1995).
11. M. Serva, G. Paladin, and J. Raboanary, *Phys. Rev. E* **52**:R9 (1996).
12. G. Paladin and M. Serva, *J. Phys. A: Math. Gen.* **29**:1381 (1996).
13. G. Paladin and M. Serva, *Phys. Rev. E* **54**:4637 (1996).
14. M. Pasquini and M. Serva, *Phys. Rev. E* **56**:2751 (1997).
15. M. Serva, *Phys. Rev. Rapid Communications* **56**:R2339 (1997).
16. R. Shankar and Ganpathy Murthy, *Phys. Rev. B* **36**:536 (1987).
17. B. Derrida, J. Vannimenus, and Y. Pomeau, *J. Phys.* **C11**:4749 (1978).
18. B. Derrida and H. J. Hilorst, *J. Phys.* **CA16**:2641 (1983).
19. J. Deutsch and G. Paladin, *Phys. Rev. Lett.* **62**:695 (1988).
20. G. Paladin and M. Serva, *Phys. Rev. Lett.* **69**:706 (1992).
21. A. Crisanti, G. Paladin, M. Serva, and A. Vulpiani, *Phys. Rev. Lett.* **71**:789 (1993).
22. A. Crisanti, G. Paladin, M. Serva, and A. Vulpiani, *J. Phys I (France)* **3**:1993 (1993).
23. A. Crisanti, G. Paladin, M. Serva, and A. Vulpiani, *Phys. Rev. E* **49**:R953 (1994).
24. M. Pasquini, G. Paladin, and M. Serva, *Phys. Rev. E* **51**:2006 (1995).
25. M. Pasquini, G. Paladin, and M. Serva, *J. Stat. Phys.* **80**:357 (1995).
26. G. Grinstein and D. Mukamel, *Phys. Rev. B* **27**:4503 (1983).
27. T. D. Schultz, D. C. Mattis, and E. H. Lieb, *Rev. Mod. Phys.* **36**:856 (1964).
28. B. M. McCoy and T. T. Wu, *Phys. Rev.* **76**:631 (1968).
29. B. M. McCoy, *Phys. Rev. B* **2**:2795 (1970).
30. B. Derrida and Y. Shnidman, *J. Phys. Lett. (France)* **45**:L-507 (1984).